Title: Pythagorean Puzzle

Brief Overview:

Students will use the Geometer's Sketchpad to construct a right triangle and discover a geometric proof of the Pythagorean Theorem. Students will test the geometric proof with acute and obtuse triangles.

NCTM 2000 Principles for School Mathematics:

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*
- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*
- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*
- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*
- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*
- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.*

Links to NCTM 2000 Standards:

- **Content Standards**

Number and Operations
Students will be able to understand meanings of operations and how they relate to one another.

Algebra
Students will be able to use mathematical models to represent and understand quantitative relationships. Students will be able to analyze change in various contexts.

Geometry
Students will be able to analyze characteristics and properties of two-dimensional geometric shapes and develop mathematical arguments about geometric relationships. Students will be able to apply transformations and use symmetry to analyze mathematical situations. Students will be able to use visualization, spatial reasoning, and geometric modeling to solve problems.

<u>Measurement</u>
Students will be able to understand measurable attributes of objects and the units, systems, and processes of measurement. Students will be able to apply techniques, tools, and formulas to determine measurements.

- **Process Standards**

<u>Problem Solving, Reasoning and Proof, Communication, Connections, and Representation</u>
These five standards are integrated throughout the unit. Students must discover the geometric proof of the Pythagorean Theorem. Students must be able to communicate the process by which the proof is demonstrated, make connections to the formula $a^2 + b^2 = c^2$, and properly construct the triangles and the associated squares on the sides of the triangles.

Links to Maryland High School Mathematics Core Learning Units:

Geometry, Measurement, and Reasoning

- **2.1.1**
The student will analyze the properties of geometric figures.

- **2.1.2**
The student will identify and verify properties of geometric figures using concepts from algebra.

- **2.1.3**
The student will use transformations to move figures, create designs, and demonstrate geometric properties.

- **2.1.4**
The student will construct, draw, and validate properties of geometric figures using appropriate tools and technology.

- **2.2.2**
The student will solve problems using two-dimensional figures and right-triangle trigonometry.

- **2.3.2**
The student will use techniques of measurement and will calculate and compare areas of two-dimensional figures and their parts.

Grade/Level:

Grades 9 – 12, Geometry.

Duration/Length:

Four 45-minute periods or two 90-minute periods.

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Use of the Pythagorean Theorem.
- Construction of segments and lines using the Geometer's Sketchpad.
- Construction of parallel and perpendicular lines using Geometer's Sketchpad.
- Calculation of lengths and areas using Geometer's Sketchpad.
- Translation of points and objects on the coordinate plane.
- Rotation of points and segments on the coordinate plane.
- Ability to solve simple algebraic equations.

Student Outcomes:

Students will:

- Construct a right triangle using Geometer's Sketchpad.
- Construct the squares on each side of the triangle using Geometer's Sketchpad.
- Perform translations using Geometer's Sketchpad.
- Conduct investigations of acute and obtuse triangles using Geometer's Sketchpad.
- Use the squares on the sides of a right triangle to geometrically support the Pythagorean Theorem.
- Develop an awareness of the relationship between the areas of the squares on the sides of a right triangle.
- Develop an awareness of the relationship of the areas of the squares on the sides of acute and obtuse triangles.

Materials/Resources/Printed Materials:

- Computer or TI-92 calculator with Geometer's Sketchpad software.
- Student Instruction Sheets, Worksheets, and Assessment.

Development/Procedures:

The teacher will begin the unit by reviewing the Pythagorean Theorem before having the students investigate this geometric proof. The teacher will also instruct students in performing translations and rotations using Geometer's Sketchpad.

Students will work in small groups to complete the constructions and Worksheet 1. After completing the constructions, and prior to completing Worksheet 2, the teacher will lead a class discussion to elicit students' awareness of the relationship between the squares on the sides of right triangles and the geometric support provided for the Pythagorean Theorem. Students will then conduct investigations of acute and obtuse triangles and complete Worksheets 3 & 4. The teacher will lead a brief class discussion concerning the relationship between the areas of the squares on the sides of acute triangles as well as the relationship between the areas of the squares on the sides of obtuse triangles. The discussion should include a comparison of the relationships between the areas of the squares on the sides of each individual type of triangle. Students choosing the Self-Guided Investigation may report on their finding to the entire class.

Assessment:

The assessment of student performance is incorporated within the activity. It assesses the students' ability to successfully complete the constructions and translations, the quality and insight of the students' conclusions about the properties of right triangles and the Pythagorean Theorem (Worksheet 1), the quality and insight of their conclusions about the relationships of the elements of the construction (Worksheet 2), as well as the quality and insight of their conclusions about acute and obtuse triangles (Worksheets 3 & 4). Answer sheets and scoring guides have been included.

Extension/Follow Up:

Students will attempt to verify the Pythagorean Theorem by dissecting the square on the *smaller* leg of the triangle. This verification is possible but requires more complicated translations and a partial dissection of the square on the larger leg as well.

Authors:

Kris Adams
Limestone Community High School
Bartonville, Illinois

Judy Stern
Home and Hospital Teacher
Montgomery Co., Maryland

John Heuer
North Hagerstown High School
Washington Co., Maryland

Pythagorean Proof Instruction Sheet

Creating a Right Triangle

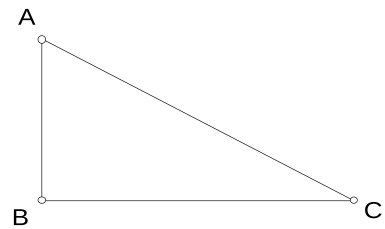
Step 1: Construct a short vertical segment *AB*.

Step 2: Construct a line through *B* perpendicular to $\overline{AB}$.

Step 3: Construct a point *C* on the line.

Step 4: Move *C* to a convenient point to create a scalene right triangle with $\overline{BC}$ greater than $\overline{AB}$.

Step 5: Hide the line. Construct $\overline{AC}$ and $\overline{BC}$.



Creating Squares on the Three Sides of the Triangle

Step 1: Mark *A* as center and rotate $\overline{AC}$ $90°$. Rename C' (the image of *C*) as *P*.

Step 2: Mark *P* as center and rotate $\overline{PA}$ $90°$. Rename A' (the image of *A*) as *Q*.

Step 3: Construct $\overline{CQ}$.

Step 4: Repeat this process on segments $\overline{BC}$ and $\overline{AB}$ to create squares *BCUT* and *ABRS*.

Before proceeding, have the teacher check your progress.

Stop here and complete Worksheet 1.

Dissecting Square BCUT

Step 1: Locate the center of square *BCUT* by constructing a diagonal and constructing the midpoint of that diagonal. Name the center of square *BCUT* point *D*.

Step 2: Hide the diagonal you constructed.

Step 3: Construct a line through *D* parallel to the hypotenuse of triangle *ABC*.

Step 4: Construct the intersection points of your parallel line with the two sides of square *BCUT*, naming point *E* on $\overline{BT}$ and point *F* on $\overline{CU}$.

Step 5: Hide the parallel line.

Step 6: Construct a line through *D* perpendicular to hypotenuse $\overline{AC}$.

Step 7: Construct the intersection points of your perpendicular line with the two sides of square *BCUT*, naming point *G* on $\overline{TU}$ and point *H* on $\overline{BC}$. Hide the perpendicular line.

Step 8: Hide the sides of square *BCUT*.

Step 9: Construct $\overline{BE}, \overline{ET}, \overline{TG}, \overline{GU}, \overline{UF}, \overline{FC}, \overline{CH}, \overline{HB}, \overline{ED}, \overline{GD}, \overline{FD},$ and $\overline{HD}$.

Step 10: Construct quadrilaterals *BEDH*, *TGDE*, *UFDG*, and *CHDF*. Shade each of these four quadrilaterals using a different color.

Before proceeding, have the teacher check your progress.

Pythagorean Proof Instruction Sheet

Translating the Quadrilaterals

Part 1:

Step 1: Mark vector $\overrightarrow{DA}$.
Step 2: Select the sides and interior of quadrilateral *CHDF* and translate.
Step 3: Similarly, translate the following quadrilaterals:

(a) *UFDG* using $\overrightarrow{DP}$.
(b) *TGDE* using $\overrightarrow{DQ}$.
(c) *BEDH* using $\overrightarrow{DC}$.

Before proceeding, have the teacher check your progress.

Part 2:

Step 1: Construct the interior of square *ABRS* and shade it with a fifth color (something different from the four previous quadrilaterals).
Step 2: Find the intersection point of two of the segments in the interior of the square on the hypotenuse and name that point *I*.
Step 3: Translate square *ABRS* into the interior of the square on the hypotenuse.

Before proceeding, have the teacher check your progress.

Investigating the Theorem

Report your findings in the following investigations on Worksheet 2.

Investigation 1: Select and drag point *C*. Referring to the relationships of the constructions discussed by the class, how do they change? How do they remain unchanged?

Investigation 2: Repeat investigation 1 by dragging point *A* and then point *B*. Describe the results.

Investigation 3: Repeat investigation 1 by dragging other points. Describe the results.

Investigation 4: Identify why this construction proves the Pythagorean Theorem.

Self-Guided Investigation

Investigation 5: The process described in this instruction sheet fails if we repeat it by dissecting the smaller square *ABRS* instead of square *BCUT*. After translating the quadrilaterals of square *ABRS* into the square on the hypotenuse, the remaining space in that square cannot contain square *BCUT* as one whole piece. Make an attempt to dissect square *BCUT* in such a way that the pieces will fit into the open region of the square on the hypotenuse and describe your method.

Non-Right Triangle Instruction Sheet

Creating an Acute Triangle

Step 1: Construct an acute scalene triangle *ABC* so that $\overline{AB}$ is the shortest side, $\overline{AC}$ is the longest side, and $\overline{BC}$ is the remaining side.
Step 2: Repeat the steps for Creating Squares on the Three Sides of the Triangle.
Stop here and complete Worksheet 3.

Creating an Obtuse Triangle

Drag point *C* to create an obtuse triangle where $\overline{AC}$ is still the longest side.
Stop here and complete Worksheet 4.

Generalizing Your Observations

After completing Worksheets 3 and 4, generalize your observations of the relationships of the sides for acute, right, and obtuse triangles. Write your observations in the space provided below.

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Name ________________ Date ________________ Period _________________

Worksheet 1: Right Triangle Measurements

Use Geometer's Sketchpad to find the measurements for 1, 3, 4, 6, 9, and 11. Check that your units are in centimeters. (Under Display, choose Preferences and set Distance Unit to cm and all precision levels to hundredths.)

1. m $\overline{AB}$ = ________________ cm.
2. (m$\overline{AB}$)^2 = ________________ cm^2.
3. The area of square *ABRS* = _________________ cm^2.
4. m $\overline{BC}$ = ________________ cm.
5. (m$\overline{BC}$)^2 = ________________ cm^2.
6. The area of square *BCUT* = _________________ cm^2.
7. (m $\overline{AB}$)^2 + (m $\overline{BC}$)^2 = ________________ cm^2.
8. The area of square ABRS + the area of square BCUT = ________cm^2.
9. m $\overline{AC}$ = ________________ cm.
10. (m$\overline{AC}$)^2 = ________________ cm^2.
11. The area of square *CAPQ* = _________________ cm^2.
12. The answer to 8 is __________________________ the answer to 11.
(less than/ equal to /greater than)

Name ________________ Date ________________ Period _________________

Worksheet 2: Investigating the Pythagorean Theorem

Investigation 1: Select and drag point *C*. Referring to the relationships of the constructions discussed by the class, how do the relationships change? How do they remain unchanged?

Investigation 2: Repeat investigation 1 by dragging point *A* and then point *B*. Describe the results.

Investigation 3: Repeat investigation 1 by dragging any other point. Describe the results.

Investigation 4: Identify why this construction proves the Pythagorean Theorem.

Name ________________ Date ________________ Period _________________

Worksheet 3: Acute Triangle Measurements

Use Geometer's Sketchpad to find the measurements for 1, 3, 4, 6, 9, and 11. Check that your units are in centimeters. (Under Display, choose Preferences and set Distance Unit to cm and all precision levels to hundredths.)

1. m $\overline{AB}$ = ________________ cm.

2. (m$\overline{AB}$)^2 = ________________ cm^2.

3. The area of square *ABRS* = _________________ cm^2.

4. m $\overline{BC}$ = ________________ cm.

5. (m$\overline{BC}$)^2 = ________________ cm^2.

6. The area of square *BCUT* = _________________ cm^2.

7. (m $\overline{AB}$)^2 + (m $\overline{BC}$)^2 = ________________ cm^2.

8. The area of square ABRS + the area of square BCUT = ________cm^2.

9. m $\overline{AC}$ = ________________ cm.

10. (m$\overline{AC}$)^2 = ________________ cm^2.

11. The area of square *CAPQ* = _________________ cm^2.

12. The answer to 8 is ___________________________ the answer to 11.
(less than/ equal to /greater than)

Name ________________ Date ________________ Period _________________

Worksheet 4: Obtuse Triangle Measurements

Use Geometer's Sketchpad to find the measurements for 1, 3, 4, 6, 9, and 11. Check that your units are in centimeters. (Under Display, choose Preferences and set Distance Unit to cm and all precision levels to hundredths.)

1. m $\overline{AB}$ = ________________ cm.
2. (m$\overline{AB}$)^2 = ________________ cm^2.
3. The area of square *ABRS* = _________________ cm^2.
4. m $\overline{BC}$ = ________________ cm.
5. (m$\overline{BC}$)^2 = ________________ cm^2.
6. The area of square *BCUT* = _________________ cm^2.
7. (m $\overline{AB}$)^2 + (m $\overline{BC}$)^2 = ________________ cm^2.
8. The area of square ABRS + the area of square BCUT = ________cm^2.
9. m $\overline{AC}$ = ________________ cm.
10. (m$\overline{AC}$)^2 = ________________ cm^2.
11. The area of square *CAPQ* = _________________ cm^2.
12. The answer to 8 is ___________________________ the answer to 11.
    (less than/ equal to /greater than)

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Scoring Guide for Teachers

Scoring Guide for Investigations

Creating Square on the Three Sides of the Triangle

3: The students correctly construct the squares of both legs of the right triangle without assistance from the instructor.

2: The students correctly construct the squares of both legs of the right triangle with minimal assistance from the instructor.

1: The students correctly construct the squares of both legs of the right triangle with considerable assistance from the instructor.

Dissecting Square *BCUT*

3: The students correctly construct the quadrilaterals of square *BCUT* without assistance from the instructor.

2: The students correctly construct the quadrilaterals of square *BCUT* with minimal assistance from the instructor.

1: The students correctly construct the quadrilaterals of square *BCUT* with considerable assistance from the instructor.

Translating the Quadrilaterals
Part 1: … Of Square *BCUT*

3: The students correctly translate the quadrilaterals into the square of the hypotenuse without assistance from the instructor.

2: The students correctly translate the quadrilaterals into the square of the hypotenuse with minimal assistance from the instructor.

1: The students correctly translate the quadrilaterals into the square of the hypotenuse with considerable assistance from the instructor.

Part 2: …Square *ABRS*

3: The students create an appropriate translation vector and translate square *ABRS* without assistance from the instructor.

2: The students create an appropriate translation vector and translate square *ABRS* with minimal assistance from the instructor.

1: The students create an appropriate translation vector and translate square *ABRS* with considerable assistance from the instructor.

Scoring Guide for Teachers

Investigating the Theorem (to be used for all investigations)

3: **The students show a thorough understanding of the proof of the Pythagorean Theorem.**
The students are able to identify the relationships in the construction that remain unchanged.
The students are able to identify which relationships change and the nature of the change.
The students are able to identify why this construction proves the Pythagorean Theorem.

2: **The students show some understanding of the proof of the Pythagorean Theorem.**
The students are able to identify some of the relationships in the construction that remain unchanged.
The students are able to identify some relationships which change and the nature of the change.
The students are partly able to identify why this construction proves the Pythagorean Theorem.

1: **The students show little understanding of the proof of the Pythagorean Theorem.**
The students are unable to identify the relationships in the construction that remain unchanged.
The students are unable to identify the relationships which change and the nature of the change.
The students are unable to identify why this construction proves the Pythagorean Theorem.

Answer Guide for Teachers

Worksheet 2: Investigating the Pythagorean Theorem

Investigation 1: Select and drag point *C*. Referring to the relationships of the constructions discussed by the class, how do they change? How do they remain unchanged?

The students should recognize that square *BCUT* changes size, as do the quadrilaterals. They should also note that the translated quadrilaterals change size with the originals. They should note that square *ABRS* remains unchanged, as does its translated image. They should also note that the translated images as a whole remain the same size as the square on the hypotenuse.

Investigation 2: Repeat investigation 1 by dragging point *A* and then point *B*. Describe the results.

The students should note similar findings as in Investigation 1, except that square ABRS will change size as will its image.

Investigation 3: Repeat investigation 1 by dragging other points. Describe the results.

The students should note that all elements of the construction move as one unit and that all sizes remain constant.

Answer Guide for Teachers

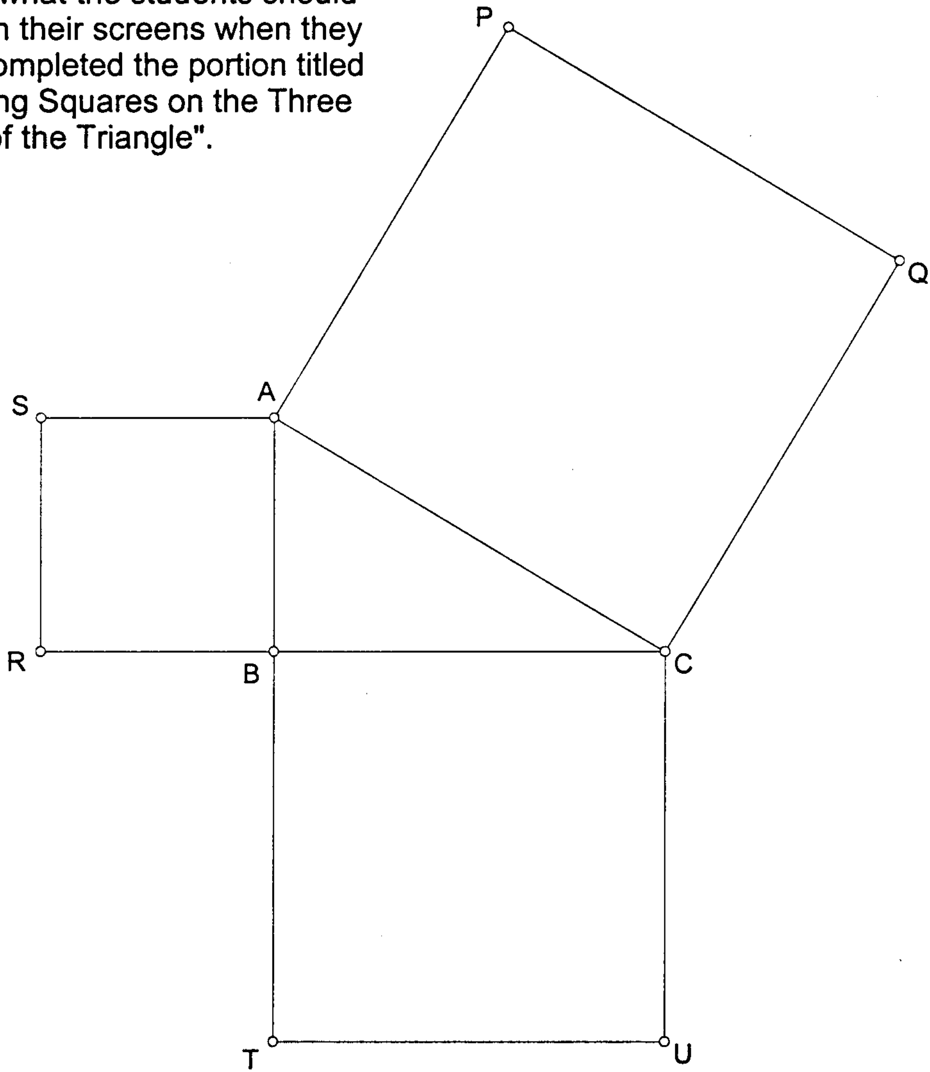
Solution to Self-Guided Investigation

The students should repeat the steps for Creating a Right Triangle and Creating Squares on the Three Sides of the Triangle. The students should divide square *ABRS* using similar processes found in "Dividing Square *BCUT*". The students should translate the quadrilaterals of square *ABRS* using processes similar to those found in "Translating the Quadrilaterals".

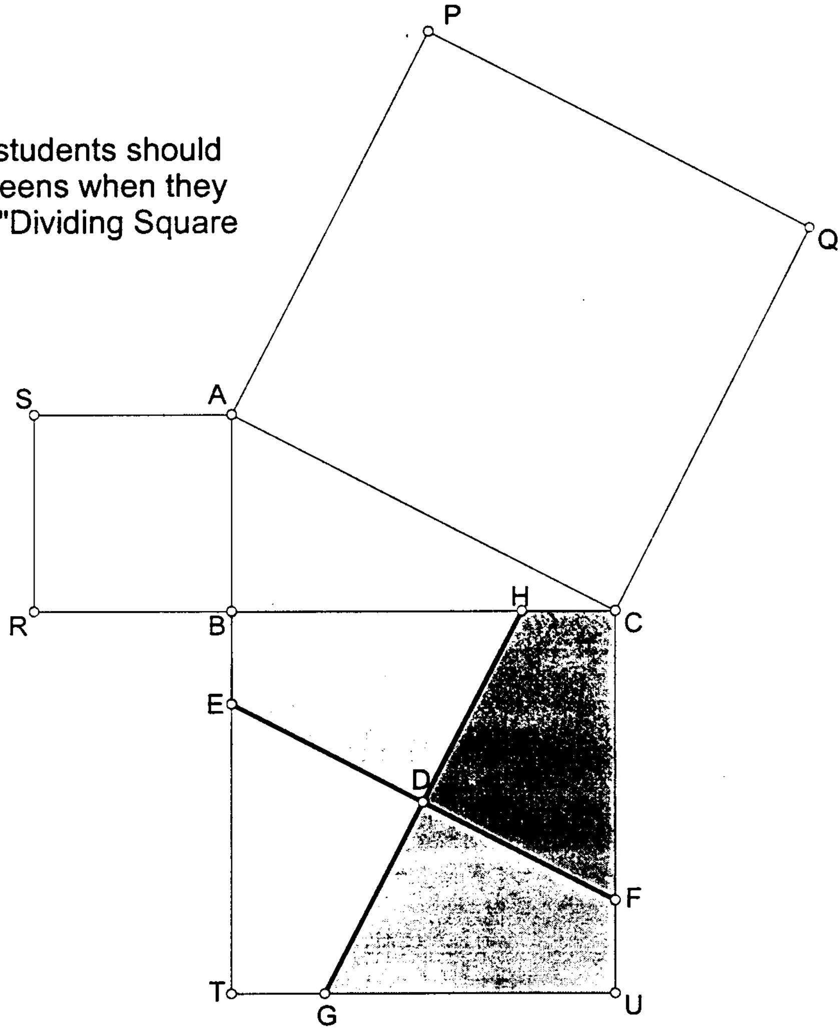
At this point the students should recognize that it is impossible to translate square *BCUT* into the open space of the square of the hypotenuse as a whole unit. The students can find a translation vector for square *BCUT* by the following method: extend the longest sides of the translated quadrilaterals located on two consecutive vertices of *APQC*; use the intersection point of these lines, which lies outside of *APQC*, as the destination of the translation vector.

At this point the students should recognize that it is necessary to construct the triangular regions located outside of *APQC* and rotate them into the square of the hypotenuse, filling the region completely.

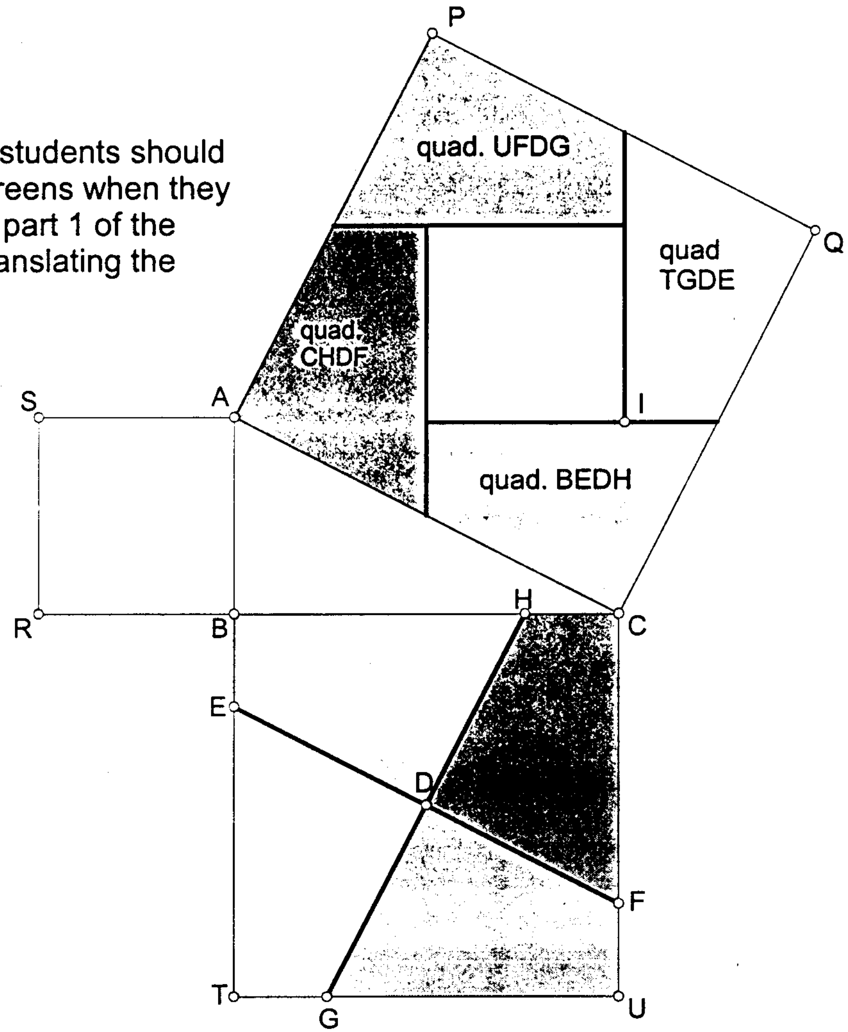
For the teacher:
This is what the students should have on their screens when they have completed the portion titled "Creating Squares on the Three Sides of the Triangle".



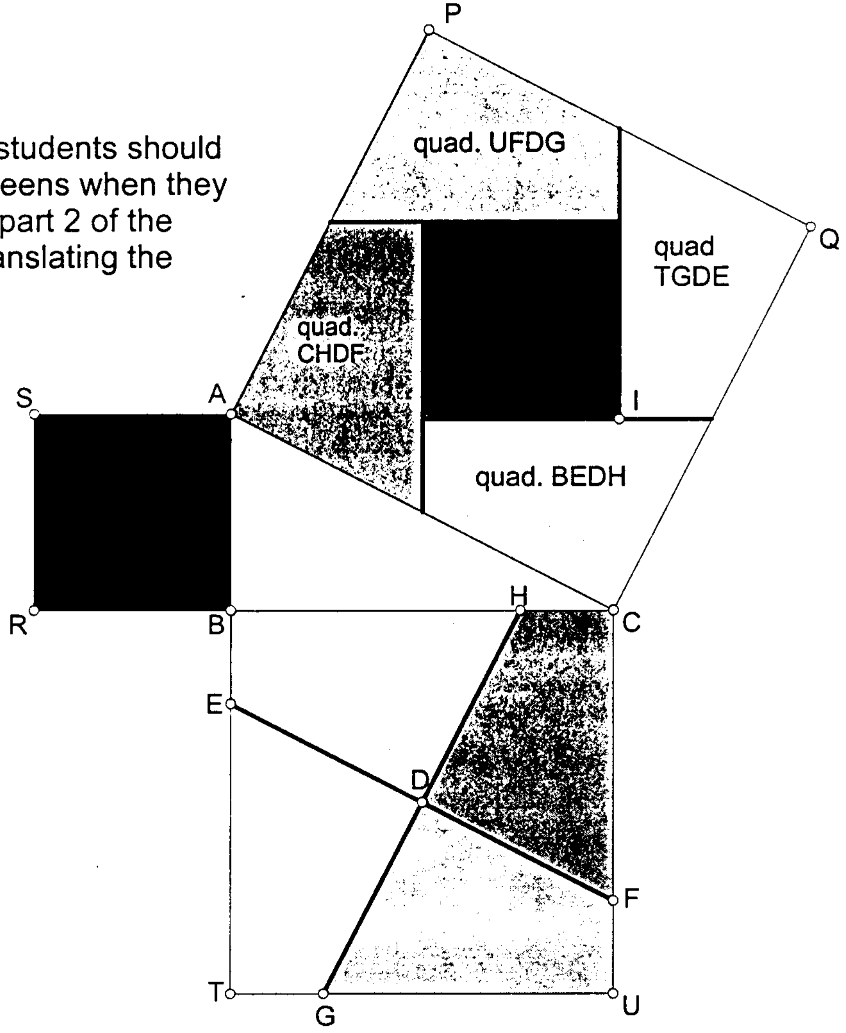
For the teacher:
This is what the students should have on their screens when they have completed "Dividing Square BCUT".



For the teacher:
This is what the students should have on their screens when they have completed part 1 of the portion titled "Translating the Quadrilaterals".



For the teacher:
This is what the students should have on their screens when they have completed part 2 of the portion titled "Translating the Quadrilaterals".



For the teacher:
This is what the students should have on their screens when they have completed the self-guided investigation.